resistance in temptation, for strength and energy when great tasks confront us, for fortitude in misfortune, for patience and submission to the inevitable ills of life.

There is but one true worship, which consists, as Christ declares, in doing the will of God, and there is but one prayer that is not heathenish, the Lord's Prayer, "Thy will be done," for it is not a beggar's petition but a vow of self-discipline. Finally, there is but one religion; it is the spread of good-will upon earth, yet it cannot be realized without an uncompromising submission to truth. Let us by all means respect the symbols in, with, and under which religious truths are taught; but the significance of a creed is more than its symbols; the meaning is greater than the dogma; the spirit is higher than the letter.

EDITOR.

ON THE DEFINITION OF AN INFINITE NUMBER.

"THE whole is greater than any one of its parts," is one of the most useful axioms of elementary geometry. In view of this fact it appears somewhat remarkable that the most useful definition of an infinite multitude should be, "The whole is equal to one of its parts." To understand this definition fully it is necessary to define the terms equal and parts.

One of the most primitive modes of proving the equality of two multitudes is the placing of the units of the multitudes in a (1, 1) correspondence. That is, if it is possible to associate with every unit of each multitude one and only one unit of the other, the two multitudes are said to be equal. The idea of a (1, 1) correspondence is doubtless one of the earliest mathematical concepts, and it is generally supposed to have given rise not only to the number concept, but also to number words.

When the child employs his fingers or other objects in counting, he gives evidence of the early development of the power "to relate things to things, or to represent a thing by a thing, an ability without which no thinking is possible."[1] Hence, there is no simpler or more definite way of proving that a part of a multitude may be equal to the whole than by establishing a (1, 1) correspondence between the units of a part and the units of the whole. The totality of positive real numbers furnishes one of the most interesting examples of such a multitude.

If in the equation

$$y = \frac{1}{x+1},$$

we let x represent any real number whatever y will also represent a real number. For two distinct values of x the corresponding vales of y will be

[1] Dedekind, *Essays on Number* (translated by Beman), page 39.

distinct. Moreover, if x is replaced by any positive real number whatsoever, y will be equal to a positive proper fraction. That is, there are just as many positive real numbers which do not exceed one as there are positive real numbers, including those which do not exceed one.

It should be observed that this is not juggling. Only perfectly valid thought processes are employed. The given equation shows in a clear and definite manner that the positive real numbers which do not exceed unity can be placed in a (1, 1) correspondence with the total number of positive real numbers. That the former constitute a part of the latter is universally accepted. Hence we must say that a part of the positive real numbers is equal to the totality of these numbers.

By employing the equation

$$2y = \frac{1}{x+1},$$

we can show, in exactly the same manner, that the positive real numbers which do not exceed one-half are equal to the totality of real numbers. More generally, by employing the equation

$$ay = \frac{1}{x+1},$$

it appears that the positive real numbers which do not exceed $\frac{1}{a}$ are equal to the totality of these numbers. It deserves to be emphasized that we are not dealing here simply with a mathematical curiosity. The real numbers furnish the foundation of a great part of the work of the student of mathematics, and it appears unpardonable to overlook any of their important properties.

The developments of mathematics have always been greatly influenced by discoveries of facts which are at variance with what was generally accepted. Of such discoveries in comparatively recent times three are especially noteworthy on account of the fundamental principles involved, namely: (1) There are perfectly consistent geometries in which the sum of the angles of a plane triangle is not equal to two right angles. (2) There are algebras in which the commutative law of multiplication does not hold. (3) There are multitudes or aggregates such that a part is equal to the whole.

The first of these discoveries is perhaps better known than either of the other two. In regard to the second, Poincaré recently said:[1] "Hamilton's quaternions give us an example of an operation which presents an almost perfect analogy with multiplication, which may be called multiplication, and yet it is not commutative, *i. e.*, the product is changed when the

Poincairé; *Bulletin des Sciences Mathématiques.* Vol. 26, 1902, page 250.

order of the factors is changed. This presents a revolution in arithmetic which is entirely similar to the one which Lobatchevski effected in geometry."

The third one of the given discoveries has perhaps made the least impression on popular thought, and yet it is certainly not less fundamental than either of the other two nor is it more foreign to the usual trend of human thought and interests. When thought is not artificially restrained it naturally enters upon the infinite, and, as in other domains of science, so in the science of the infinite, nothing capable of proof ought to be passed by without proof. Moreover, it is most important that the facts which do not agree with what has been generally accepted should be popularized, on account of their corrective influence on the human intellect.

In the above proof that a part of the real positive numbers is equal to the totality of the numbers, the term part was employed in its universally accepted sense. In general, we shall say that a given aggregate is a part of a second aggregate, provided all the units of the first aggregate are contained in the second and the second contains at least one unit which is not in the first. For instance, we shall say that the even positive numbers are a *part* of all the natural numbers. Since it is clearly possible to associate with each natural number its double, the natural numbers constitute an infinite aggregate. In fact, we may establish a (1, 1) correspondence between the natural numbers and a part of these numbers in an unlimited number of different ways. For instance, it is possible to associate any multiple or any power of the number with each of the natural numbers.

By the same method it may be proved that time, according to the common conception, is infinite; for there is a (1, 1) correspondence between the total number of hours and the total number of half hours. If the half hours from any period would be denoted by the natural numbers, the hours would correspond to the even numbers. Similarly, it may be observed that space (defined analytically by coördinates) is infinite, according to the given definition.

If it is possible to establish a (1, 1) correspondence between two infinite totalities, they are said to be of the same power. The totality (M_0) formed by the natural numbers is especially important. Any totality which has the same power as M_0 is said to be countable. It is an extremely interesting fact that all algebraic numbers are countable. In particular, it is possible to establish a (1, 1) correspondence between all rational numbers and the natural numbers. The developments along this line belong to the important subject known as the theory of aggregates (ensemble, Mengenlehre), where the full significance of the use of the infinite in mathematics is exhibited.

The simplicity of the given definition may perhaps become more evident if the matter is stated as follows: If a totality (M) of units is given, only two cases are possible. Either M contains a part which contains as many

units as M or no such part exists in M. In the former case, M is said to be infinite; in the latter it is finite. It is not difficult to show that this definition does not violate the ordinary conception that the infinite is unlimited, while the finite is limited.

Dedekind was the first (1880) to use this definition of an infinite aggregate or totality as the foundation to establish the science of numbers. Bolzano and others pointed out at an earlier date that infinite aggregates have the property which is here used as a definition. It is now very generally employed, and the mind that can dwell upon it long enough to grasp even the most direct bearings cannot fail to derive from it an unusual amount of pleasure and profit, such as only great thoughts can give.

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NOTE ON "A BUDDHIST GENESIS."

Since my translation of the Buddhist Genesis document appeared in the January *Monist,* I have found that Rockhill rendered it from the Tibetan in 1884. (*Life of the Buddha and Early History of His Order.* Translated from the Tibetan. By W. Woodville Rockhill. London (Trübner's Oriental Series, 1884). I had known this book for years, but it escaped me when making the Genesis translation and also in my *Buddhist Bibliography* (London, 1903). In Rockhill's volume the Genesis document comes at the very beginning. Like the Sanskritised Prâkrit version used by me, it belongs to the Vinaya Pitaka. The Tibetan Canon is that of the sect of Realists (*Sarvastivada*), whose account of the compilation of the Scriptures was translated by Suzuki, also in the January *Monist.* There are two versions of the Genesis document in the Tibetan Vinaya Pitaka: A short one in the Vinaya-vastu (corresponding in part to the Pâli Mahâvaggo), and a long one (translated by Rockhill) in the Vinaya-Vibhâga (Pâli Bhikkhu-Vibhanga). The Theravâda sect, who have handed down the Pâli Tripitaka, do not place this document in the Vinaya, but in the Sûtra Pitaka. Thus do we prove the truth of the Island Chronicle of Ceylon, which says that the Realists, the Great Council, and many other sects, made recensions of the Canon to suit themselves. We must never forget that the Pâli, though the oldest version of the Canon known, is by no means the only one. The Mahâsamghika (Great Council) school also claims to be the oldest, and their Book of Discipline has come down to us in a fifth-century Chinese translation. Suzuki also gave us extracts from this, and we saw therefrom that they had no Abhidharma. This looks as if their Canon belonged to an earlier period than the Pâli, for the Abhidharma was in the nature of commentary, and was compiled after the Buddhists had split up into sects. We